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SHRINKING BASES AND BANACH SPACES Z^{**}/Z

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ABSTRACT

Some results on shrinking bases and Banach spaces of type Z^{**}/Z are given in this paper. As a consequence, it is proved that a separable Banach space X is isometric to a space Y^{**}/Y where Y^* has a shrinking basis.

The vector spaces used here are defined on the field K of real or complex numbers. Our notation is standard. By the term "subspace" we always mean a linear subspace. Given a subspace Y of a Banach space X, $\tilde{Y}$ will denote the weak-star closure of Y in X^{**}. We shall use $\langle \cdot,\cdot \rangle$ for the bilinear form of a duality. A Banach space X is *quasireflexive* if X^{**}/X has finite dimension. It is well known [2] that there are quasireflexive non-reflexive Banach spaces.

1. Shrinking bases

We shall use in the sequel the following two results which can be found in [7] and [8] respectively:

(a) *Let Z be a separable Banach space. Let (z_{mn}) be a double sequence in Z^* such that $(z_{mn})_{n=1}^{\infty}$ converges to the origin in the weak-star topology and*

$$0 < \inf\{ \| z_{mn} \| : n = 1, 2, \ldots\}, \qquad m = 1, 2, \ldots .$$

Then there is a subsequence $(y_{mn})_{n=1}^{\infty}$ of $(z_{mn})_{n=1}^{\infty}$, $m = 1, 2, \ldots$, such that

$$y_{11}, y_{12}, y_{21}, \ldots, y_{1n}, y_{2(n-1)}, \ldots, y_{(n-1)2}, y_{n1}, \ldots \tag{1}$$

is a weak-star basic sequence in Z^*. Moreover, if Z is separable, (1) *can be chosen to be boundedly complete.*

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(b) *Let Y be a closed subspace of a Banach space X. Then $X + \tilde{Y}$ is a closed subspace of X^{**}.*

The result quoted as (a) is a double-sequence version of a theorem due to W. B. Johnson and H. P. Rosenthal [4].

THEOREM 1. *Let X be an infinite-dimensional Banach space such that X^* is separable. Let T be a closed separable subspace of X^{**} containing X. Then there is a closed subspace Y of X with the following properties*:

(1) *Y has a shrinking basis.*

(2) *$X + \tilde{Y}$ contains T.*

PROOF. In the case $X = T$ it is enough to select a shrinking basic sequence in X [1] and define Y as its closed linear hull. Let us suppose now that X and T do not coincide. In this case we can choose a sequence (z_n) in $T \sim X$ such that its closed linear hull coincides with T. For every positive integer m we select a sequence (t_{mn}) of non-zero elements in X which weak-star converges to z_m. Obviously

$$0 < \inf\{ \| t_{mn} - z_m \| : n = 1, 2, \ldots\}$$

and

$$\lim_n \langle t_{mn} - z_m, u \rangle = 0, \qquad \text{for every } u \in X^*,$$

hence we can apply result (a) to X^* obtain a subsequence $(y_{mn})_{n=1}^{\infty}$ of $(t_{mn})_{n=1}^{\infty}$ such that

$$\begin{gathered} y_{11} - z_1, y_{12} - z_1, y_{21} - z_2, \ldots, y_{1n} - z_1, y_{2(n-1)} - z_2, \ldots, \\ y_{(n-1)2} - z_{n-1}, y_{n1} - z_n, \ldots . \end{gathered} \tag{2}$$

is a weak-star basic sequence in X^{**}. Let S be the weak-star closed linear hull of (2) in X^{**}, $Y := S \cap X$. Given two positive integers p, q,

$$(y_{pq} - z_p) - (y_{pn} - z_p) = y_{pq} - y_{pn} \in Y \tag{3}$$

for every positive integer n. Letting n converge to ∞ we get

$$y_{pq} - z_p \in \tilde{Y}$$

hence S coincides with $\tilde{Y}$. If $Y^{\perp}$ denotes the subspace of X^* orthogonal to Y, $X^*/Y^{\perp}$ has a basis (and this space can be identified with Y^*) hence Y has a shrinking basis, after a result of W. B. Johson, H. P. Rosenthal and M. Zippin [5]. Finally

$$z_m \in X + \tilde{Y}, \qquad m = 1, 2, \ldots,$$

then T is contained in $X + \tilde{Y}$ in view of result (b). q.e.d.

The following corollary is a straightforward consequence of the former theorem:

COROLLARY 1.1. *An infinite-dimensional Banach space X such that X^* is separable is quasireflexive if and only if every closed subspace of X with a shrinking basis is quasireflexive.*

We proved the following result in [8]:

(c) *Let X be an infinite-dimensional Banach space such that X^{**} is separable. Let T be a closed subspace of X^{**} containing X. Then there is an infinite-dimensional closed subspace Y of X such that $X + \tilde{Y} = T$.*

THEOREM 2. *Let X be an infinite-dimensional Banach space such that X^{**} is separable. Let T be a closed subspace of X^{**} containing X. Then there exists a closed subspace Y of X such that*

(1) *Both Y and Y^* have shrinking bases,*

(2) $X + \tilde{Y} = T$.

PROOF. In the case $T = X$ the result (c) gives an infinite-dimensional reflexive subspace Z of X. We can take as Y a subspace of Z with basis. (In [4] it is proved for the first time that every infinite-dimensional Banach space X such that X^{**} is separable has infinite-dimensional reflexive subspaces.)

Let us suppose that $T \neq X$. Again by the result (c) we can take a closed subspace Z of X such that $Y + \tilde{Z} = T$. Let (z_m) be a total sequence in $\tilde{Z}$ such that

$$z_m \notin Z, \qquad m = 1, 2, \ldots .$$

Given a positive integer m we can choose a sequence (t_{mn}) of non-zero elements in Z converging to z_m in the weak-star topology of X^{**}. Proceeding as in the proof of Theorem 1 we can select a subsequence $(y_{mn})_{n=1}^{\infty}$ of (t_{mn}) such that (2) (cf. the proof of Theorem 1) is a weak-star basic sequence in X^{**}. Since Z is separable, we can even choose (2) to be boundedly complete. If S denote the weak-star closed linear hull of (2) in X^{**}, let $Y := S \cap X$.

In both cases Y and Y^* have shrinking bases and $X + \tilde{Y}$ coincides with T. q.e.d.

COROLLARY 1.2. *Let V be a separable Banach space such that V^{**}/V is* *isomorphic to* $C[0, 1]$. *Then, for any separable Banach space* X, *there exists a closed subspace* Y *of* V *with the following properties*:

(1) *Both* Y *and* Y^* *have shrinking bases.*

(2) Y^{**}/Y *is isomorphic to* X.

PROOF. Let ζ be the canonical mapping from V^{**} onto V^{**}/V. Let Z be a subspace of V^{**}/V isomorphic to X. Let $T := \zeta^{-1}(Z)$. Theorem 2 gives a closed subspace Y of V such that both Y and Y^* have shrinking bases and $V + \tilde{Y} = T$. Let φ and ψ be the canonical mappings from $\tilde{Y}$ onto $\tilde{Y}/Y$ and from T onto T/V respectively. Given $\tilde{y} \in \tilde{Y}/Y$ let y be an element of $\tilde{Y}$ such that $\varphi(y) = \tilde{y}$. Defining $f(\tilde{y}) = \psi(y)$, it is obvious that f is an isomorphism from $\tilde{Y}/Y$ onto T/V. Then we reach the conclusion observing that $\tilde{Y}/Y$ is isomorphic to Y^{**}/Y and T/V is isomorphic to X. q.e.d.

2. Separable Banach spaces Z^{**}/Z

J. Lindestrauss [6], following some work done by R. C. James [3], proved that if X is a separable Banach space, there exists another separable Banach space Z such that a Z^{**}/Z is isomorphic to X and Z^* has a shrinking basis. Here we shall prove that Z can be chosen in such a way that Z^{**}/Z is even isometric to X. We shall prove also that there is a Banach space L with L^{**} separable which is universal for the class of all separable Banach spaces X in the following sense: given a separable Banach space X, there exists a closed subspace Y of L such that $(L/Y)^{**}/(L/Y)$ is isometric to X.

PROPOSITION 1. *Let* Z *be an infinite-dimensional Banach space such that* Z^{**} *is separable. Let* T *be a closed subspace of* Z^{**} *such that* $Z \subset T$. *Then there exists a closed subspace* Y *of* Z *with the following properties*:

(1) Z/Y *has a shrinking basis.*

(2) $(Z/Y)^{**}/(Z/Y)$ *is isometric to* Z^{**}/T.

PROOF. Let us consider first the case $T = Z^{**}$. Let F be a reflexive subspace of Z^* with basis [4] and let Y be the subspace of Z orthogonal to F. Then Z/Y has a shrinking basis and $Z + \tilde{Y} = T$. To complete the proof let us suppose now $T \neq Z^{**}$. We shall denote by $T^{\perp}$ the subspace of Z^{***} orthogonal to T. Let

$$\{z_1, z_2, \ldots, z_n, \ldots\}$$

be a weak-star total subset of $T^{\perp}$ such that $\| z_n \| = 1$, $n = 1, 2, \ldots$. Now result (c) gives a closed subpsace U of Z such that $Z + \tilde{U} = T$. Let P be the subspace of Z^* orthogonal to U. For any positive integer m we can find a sequence (t_{mn}) in P, with $\| t_{mn} \| \neq 0$, $n = 1, 2, \ldots$, which converges to z_m in the weak-star topology of Z^{***}.

It follows that

$$\lim_n \langle z, t_{mn} \rangle = 0, \qquad z \in Z, \quad m = 1, 2, \ldots,$$

hence, using the result (a) and the fact that Z^* is separable, we can get a subsequence $(y_{mn})_{n=1}^{\infty}$ of $(t_{mn})_{n=1}^{\infty}$, $m = 1, 2, \ldots$, such that

$$(4) \qquad y_{11}, y_{12}, y_{21}, \ldots, y_{1n}, y_{2(n-1)}, \ldots, y_{(n-1)2}, y_{n1}, \ldots$$

is a boundedly complete basis sequence. Thus, if Y denotes the subspace of Z orthogonal to the set consisting of the vectors in (4), Z/Y has a shrinking basis and $\tilde{Y}$ is contained in T (this follows from the fact that z_n is a cluster point of (4) in Z^{***} equipped with the weak-star topology, $n = 1, 2, \ldots$). Moreover, $\tilde{Y}$ contains $\tilde{U}$ and then $Z + \tilde{Y} = T$.

Let $Y^{\perp}$ be the subspace of Z^* orthogonal to Y. Then $\tilde{Y}$ is the subspace of Z^* orthogonal to $Y^{\perp}$. Given an arbitrary element u of $(Z/Y)^{**}$, let us denote by $\psi(u)$ the linear manifold in Z^{**} whose elements restricted to $Y^{\perp}$ coincide with u. $\psi(u)$ is an element of $Z^{**}/\tilde{Y}$ and, in the light of the Hahn–Banach theorem, ψ is an isometry from $(z/Y)^{**}$ onto $Z^{**}/\tilde{Y}$. It follows that

$$\psi(Z/Y) = T/\tilde{Y},$$

hence $(Z^{**}/\tilde{Y})/(T/\tilde{Y})$ is isometric to $(Z/Y)^{**}/(Z/Y)$. Finally, since $\tilde{Y}$ is contained in T, Z^{**}/T is isometric to $(Z^{**}/\tilde{Y})/(T/\tilde{Y})$. q.e.d.

In [3, Thm. 1], R. C. James proves that if X is a Banach space with a shrinking basis, there exists a separable Banach space Z having a topological complement Z_1 in Z^{**} such that Z_1 is isometric to X^*. Moreover, the corresponding projection mapping from Z^{**} onto Z_1 has norm equal to one. Using this result we can fix a separable Banach space L having a topological complement S in L^{**} such that S is isometric to l^1 and the corresponding projection mapping from L^{**} onto S has a norm equal to one.

THEOREM 3. *Given a separable Banach space X, there exists in L a closed subspace Y with the following properties*:

(1) *L/Y has a shrinking basis.*

(2) *$(L/Y)^{**}/(L/Y)$ is isometric to x.*

PROOF. The subspace S of L is isometric to l^1, hence we can find a closed subspace H of S such that S/H is isometric to X. Let

$$T := L + H.$$

Then T is a closed subspace of L^{**} which contains L. Proposition 1 applies and we get a closed subspace Y of L such that L/Y has a shrinking basis and $(L/Y)^{**}/(L/Y)$ is isometric to L^{**}/T.

Let $\varphi(\psi)$ be the canonical mapping from L^{**} onto L^{**}/T (from S onto S/H). Given an arbitrary element u in L^{**}/T we can find x in L^{**} such that $\varphi(x) = u$. Let x_1 be the projection of x onto S along L. Let

$$\zeta(u) = \psi(x_1).$$

Then it is easy to prove that ζ is an isometry from L^{**}/T onto S/H. Therefore $(L/Y)^*/(L/Y)$ is isometric to X. q.e.d.

PROPOSITION 2. *Let X be an infinite-dimensional Banach space such that X^{**} is separable. Then there exists a closed subspace Y of X with the following properties*:

(1) *Both Y and Y^* have shrinking basis.*

(2) *Y^{**}/Y is isometric to X^{**}/X.*

PROOF. In the case that X is reflexive, it is enough to choose as Y a closed subspace Y of X having a basis. Otherwise let B be the closed unit ball of X. Let $\| \cdot \|$ be the norm both in X^{**} and in X^{**}/X. Let φ be the canonical mapping from X^{**} onto X^{**}/X. Denoting by A the interior of $\varphi(\tilde{B})$ in X^{**}/X, we choose in A a dense countable subset M such that $0 \notin M$. Given $x \in M$ and a positive integer n, let us choose a vector $y(x, n)$ in X^{**}/X such that

$$\| y(x, n) \| < \| x \| + 1/n, \qquad \varphi(y(x, n)) = x.$$

The elements in

$$\{ y(x, n) : x \in M, n = 1, 2, \ldots \}$$

can be ordered in a single sequence (z_n). Given a positive integer m we can find a sequence (t_{mn}) in Z converging to z_m in the weak-star topology of Z^{**} and such that

$$\| t_{mn} \| = \| z_m \|, \qquad n = 1, 2, \ldots . \tag{5}$$

Proceeding as in the proof of Theorem 1 we can find a subsequence $(y_{mn})_{n=1}^{\infty}$ of $(t_{mn})_{n=1}^{\infty}$, $n = 1, 2, \ldots$, such that the sequence (2) becomes a weak-star basic sequence in X^{**}. Since X^{**} is separable, we can choose (y_{mn}) such that (2) is boundedly complete. Let S be the weak-star closed linear hull of (2) in X^{**}, and let $Y := S \cap X$. Then $\tilde{Y}$ coincides with S. Denoting by $Y^{\perp}$ the subspace of X^* orthogonal to Y, we can identify Y^* with $X^*/Y^{\perp}$ as usual. It follows that Y^* has a shrinking basis, hence, by a result in [5], Y has also a shrinking basis. Moreover

$$X \cup \{z_m : m = 1, 2, \ldots\}$$

is a total subset of X^{**} hence, by result (b), $X + \tilde{Y}$ coincides with X^{**}.

Let $|||\cdot|||$ be the gauge of $\widetilde{B \cap Y}$ in $\tilde{Y}$. We shall assume Y endowed with the norm $|||\cdot|||$. The corresponding norm in $\tilde{Y}/Y$ shall be denoted by $|||\cdot|||$ also. The usual identification of Y^* and $X^*/Y^{\perp}$ makes $\tilde{Y}$ and Y^* isometrically isomorphic, hence Y^{**}/Y is also isometrically isomorphic to $\tilde{Y}/Y$.

Let ψ be the canonical mapping from $\tilde{Y}$ onto $\tilde{Y}/Y$. Given an arbitrary element x in X^{**}/X let u be an element in X^{**} such that $\varphi(u) = x$. Let

$$u = u_1 + u_2, \qquad u_1 \in X, \quad u_2 \in \tilde{Y}, \quad \zeta(u) = \psi(u_2).$$

It is easy to prove that ζ is a linear injective mapping from X^{**}/X onto $\tilde{Y}/Y$. Let y be an element in $\tilde{Y}/Y$ such that $|||y||| < 1$. We can find an element v in $\tilde{Y}$ such that simultaneously $|||v||| < 1$ and $\psi(v) = y$. It is obvious that $|||v||| > \|v\|$. Hence

$$\|\zeta^{-1}(y)\| = \|\varphi(v)\| \leqq \|v\| \leqq |||v||| < 1.$$

Therefore, denoting by P the interior of $\psi(\widetilde{B \cap Y})$ in $\tilde{Y}/Y$, it follows that

$$\zeta(A) \supset P.$$

Take x to be an arbitrary element of M. We can find a positive integer n such that

$$\|x\| + 1/n < 1.$$

Another positive integer m can be determined in such a way that

$$\|z_m\| < \|x\| + 1/n < 1, \qquad \varphi(z_m) = x. \tag{6}$$

Since z_m belongs to $\tilde{Y}$, it follows from (5) and (6) that

$$|||\zeta(x)||| = |||\psi(z_m)||| < |||z_m||| < 1$$

hence

$$\zeta(A) \subset P.$$

Then ζ is an isometry from X^{**}/X onto $\tilde{Y}/Y$. q.e.d.

THEOREM 4. *Given a separable Banach space X there exists a Banach space Y with the following properties*:

(1) *Both Y and Y^* have shrinking basis.*

(2) *X is isometric to Y^{**}/Y.*

PROOF. By Theorem 3 there exists an infinite-dimensional separable Banach space Z such that Z^{**}/Z is isometric to X. By Proposition 2 there exists a closed subspace Y of Z such that Y and Y^* have shrinking bases and Y^{**}/Y is isometric to X. q.e.d.

REFERENCES

1. D. W. Dean, I. Singer and L. Sterbach, *On shrinking basic sequences in Banach spaces*, Studia Math. **40** (1971), 23–33.

2. R. C. James, *Bases and reflexivity of Banach spaces*, Ann. of Math. **52** (1950), 518–527.

3. R. C. James, *Separate conjugate spaces*, Pacific J. Math. **10** (1960), 563–571.

4. W. B. Johnson and H. P. Rosenthal, *On w*-basic sequences and their applications to the theory of Banach spaces*, Studia Math. **43** (1972), 77–79.

5. W. B. Johnson, H. P. Rosenthal and W. Zippin, *On bases, finite-dimensional decompositions and weaker structures in Banach spaces*, Isr. J. Math. **9** (1971), 488–506.

6. J. Lindenstrauss, *On Jame's paper "separate conjugate spaces"*, Isr. J. Math. **9** (1971), 274–284.

7. M. Valdivia, *Bases y casirreflexividad en espacios de Banach*, Rev. Real Acad. Cienc. Exactas, Físicas y Naturales, Madrid, to appear.

8. M. Valdivia, *Banach spaces X with X^{**} separable*, Isr. J. Math. **59** (1987), 107–111.